Hybrid Overset/Cartesian Grid Methods for Incompressible Flow

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Scientific Computing

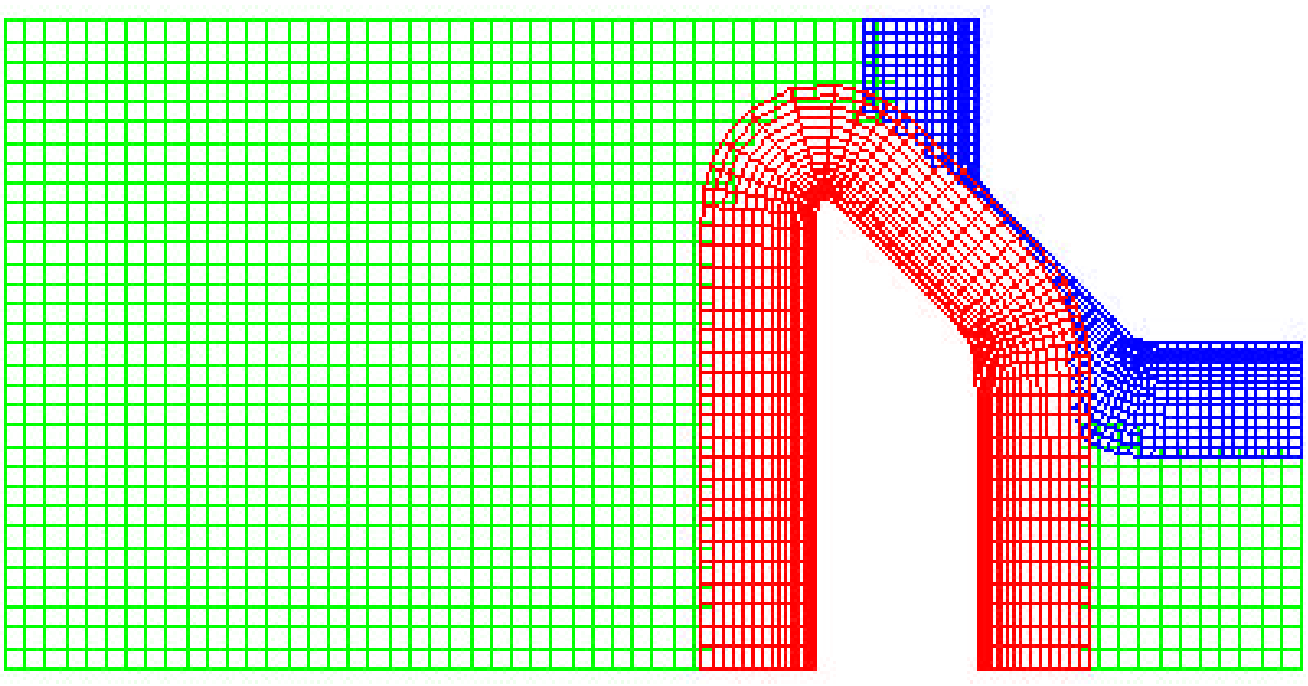
Outline

- Grid Taxonomy
- Grid Generation
- Incompressible Flow and Projection Algorithms
- Advection
- Elliptic Solver Algorithm
- Work in Progress

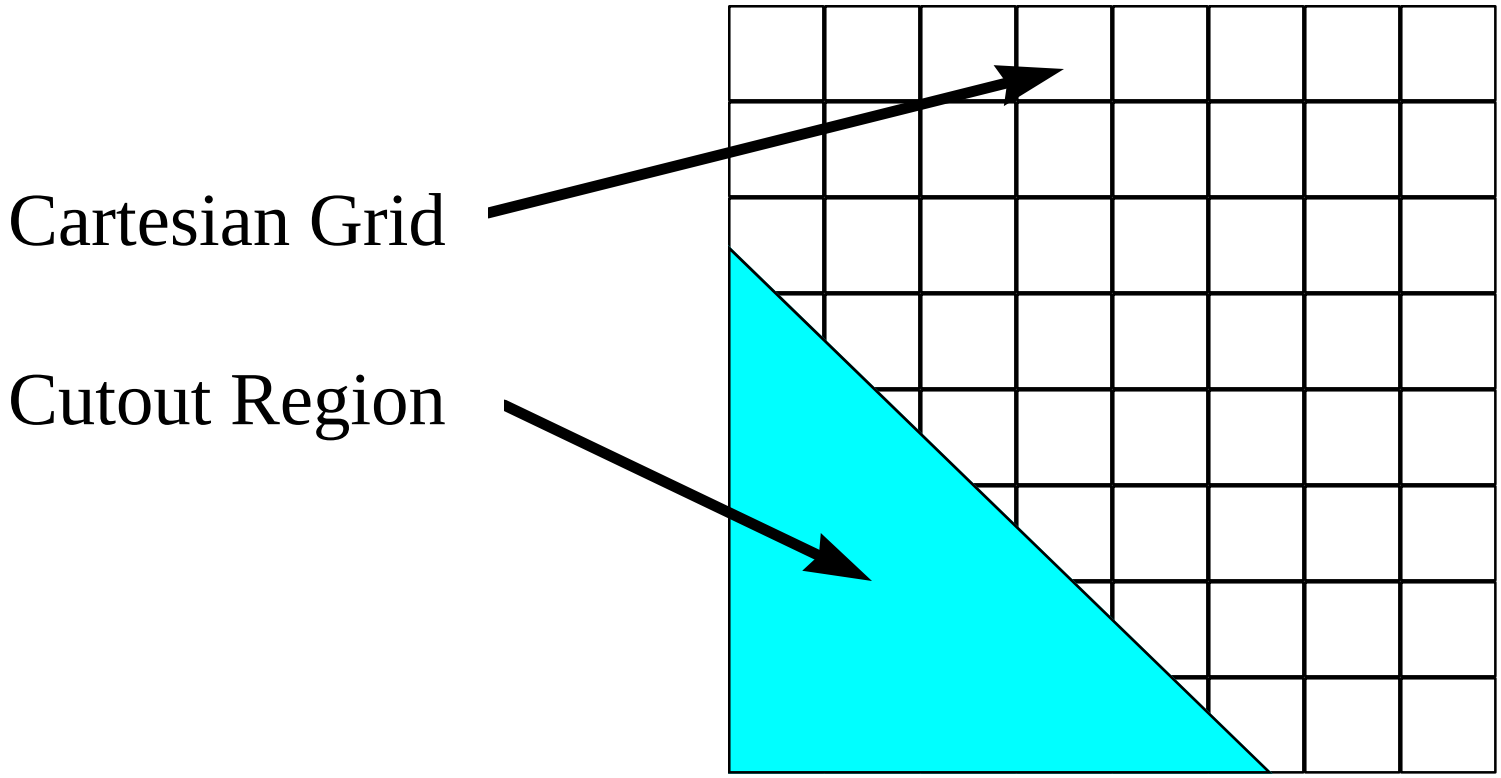
Relevant Work

- Chesshire/Henshaw
  - Conservation on Composite Grids (SIAM J. Sci. Stat. Comp. **15**, No. 4, 1994)
  - Mutligrid Method (SIAM J. Sci. Stat. Comp. **8**, No. 6, 1987)
- Colella/Johansen
  - Cartesian Elliptic Solver (Submitted to SIAM J. Sci. Stat. Comp.)
- Almgren/Bell/Colella/Marthaler
  - Incompressible Cartesian Method (AIAA Paper 95-1743, 12th CFD Proceedings, San Diego, 1995)

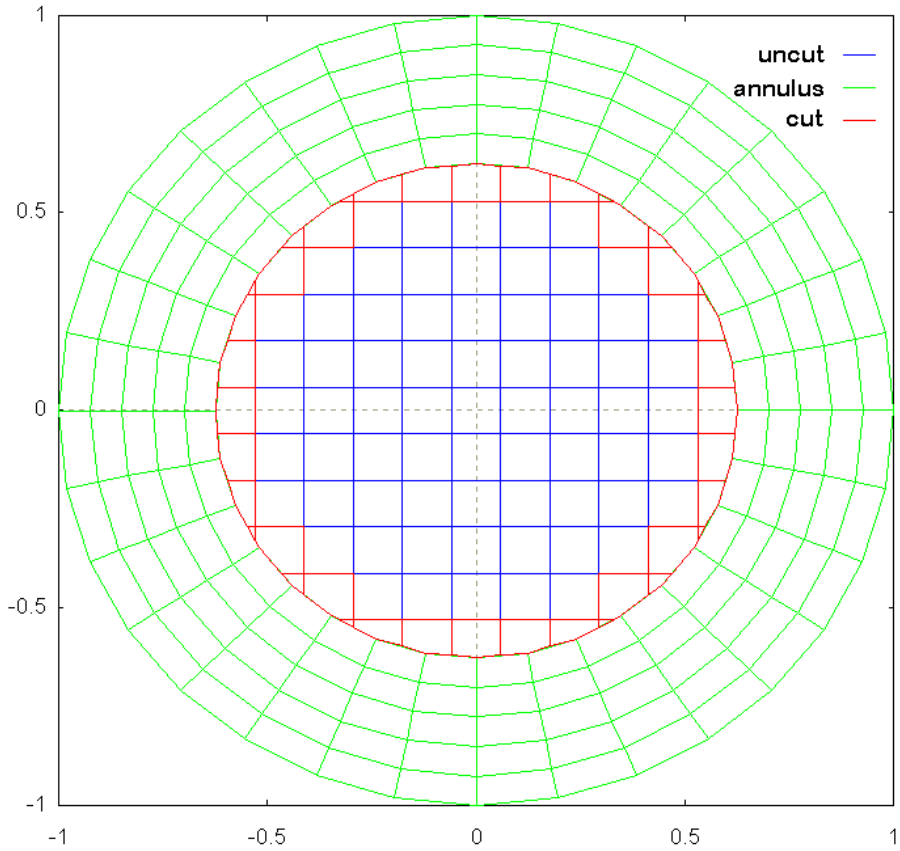
Grid Taxonomy I: Overset Grids



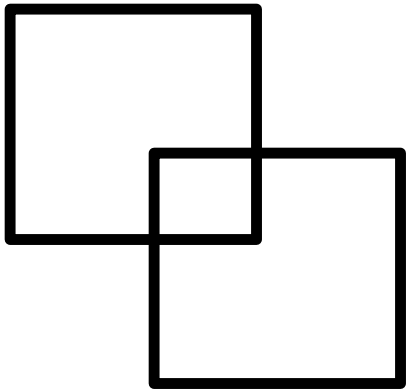
Grid Taxonomy II: Cartesian Grids



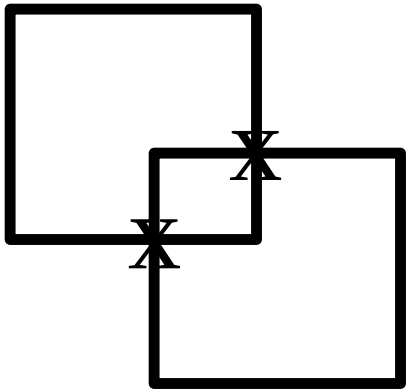
Hybrid Grid



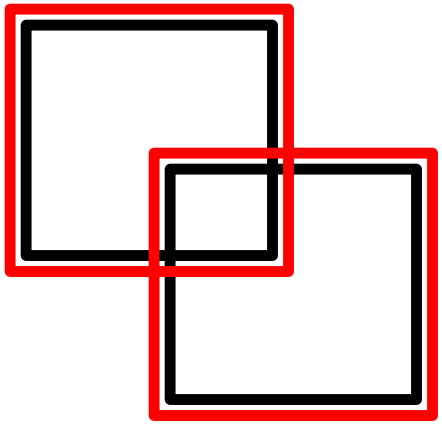
Weiler Algorithm



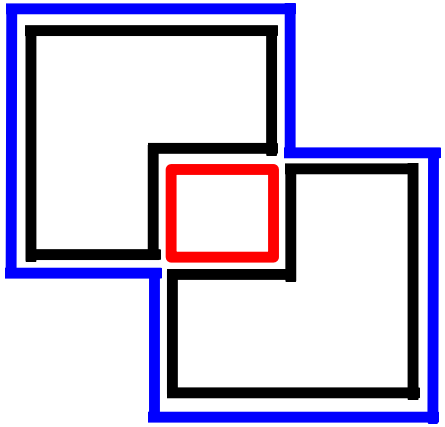
1) Intial Polygons



2) Find Intersections



3) Duplicate Contours



4) Reconnect Contours

Incompressible Flow

- The prototypical equations for low Mach number flow are the incompressible Navier-Stokes equations.

- The equations are discretized in a variety of ways. They all share incompressibility constraints that must be satisfied. The incompressibility maintained by continually projecting the velocity onto the space of divergence free vectors.

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \bullet \nabla)\vec{u} + \nabla p - \nu \Delta \vec{u} - \vec{f} = 0$$

$$\nabla \bullet \vec{u} = 0$$

$$B(\vec{u}, p) = 0, \quad \text{(On the boundary)}$$

$$u(x,0) = u_0(x), \quad \text{(Initial conditions)}$$

$\vec{u}$ velocity

p pressure

ν viscosity

Hodge Decomposition

- Hodge Formulas

$$\vec{u} = \vec{u}_d + \nabla\phi$$

taking the divergence:

$$\nabla \bullet \vec{u} = \Delta\phi$$

$$\vec{u}_d = \vec{u} - \nabla\phi$$

define the operators:

$$Q = \nabla(\Delta^{-1}(\nabla \bullet \ ))$$

$$P = I - Q$$

Projection Algorithm

- Predictor-Corrector Style Solution Advance for

$$U_t + U \bullet \nabla U = -\nabla p, \quad \nabla \bullet U = 0$$

- Predictor
  - Advance velocities from the cell centers to the cell edges at a half time level by taylor explansion. Use upwinding to resolve waves.
  - Use an edge centered numerical projection to insure that the upwinded velocities are divergence free.
  - Form the advective term and use a lagged pressure gradient to compute a time and space centered change in velocity.

$$(\delta U / \Delta t)^* = -(U \bullet \nabla U)^{n+1/2} - (\nabla p)^{n-1/2}$$

Projection Algorithm (cont.)

- Corrector
  - Project the change in velocity to advance the velocity

$$U^{n+1} = U^{n} + \Delta t\, P(\delta U / \Delta t)^{*}$$

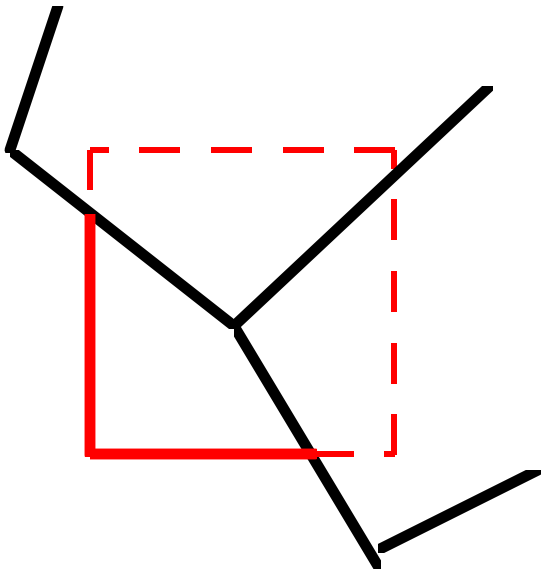
  - Advance the pressure gradient using the potential part of the decomposition.

$$(\nabla p)^{n+1/2} = (\nabla p)^{n-1/2} + (I - P)(\delta U / \Delta t)^{*}$$

Cut Cell Advection Difficulties

- The cut cells within the hybrid grid can become small. This can limit the time step if explicit advection approximations are used.
- For cut cells a scheme is derived that is
  - Conservative
  - Approaches a reasonable value as the cut cell volume goes to zero.
  - Approaches the uncut cell finite volume cell formulation as the volume fraction approaches unity.

Cut Cell Update



Conservative Unstable Scheme

$$U^* = U - \frac{\Delta t}{\Lambda V} \sum F_{cut}$$

Stable reference value:

$$\tilde{U} = U - \frac{\Delta t}{V} \sum F_{uncut}$$

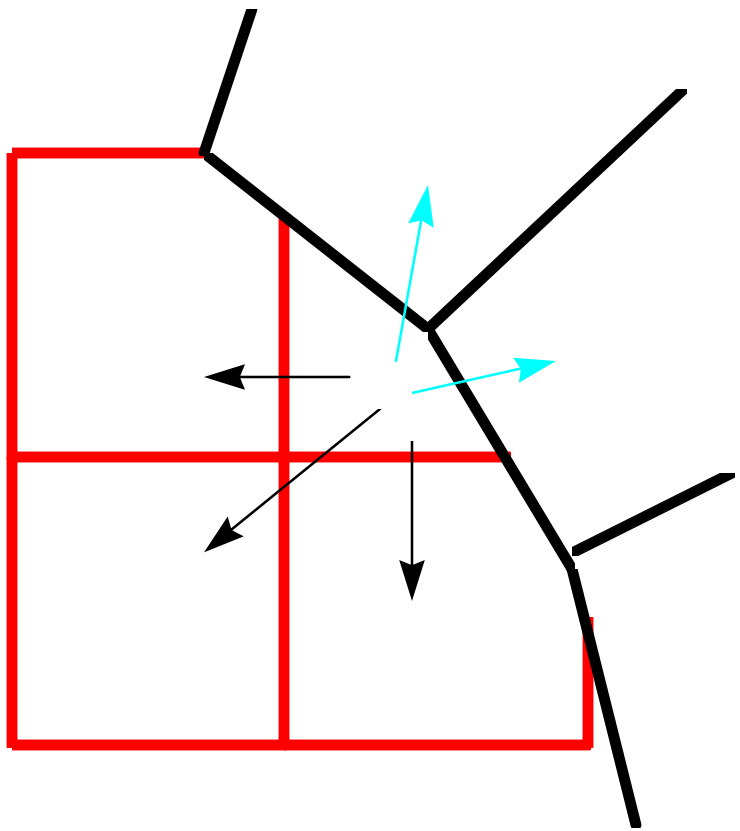
Stable Scheme:

$$\hat{U} = U - \frac{\Delta t}{V} \sum F_{cut} - (1-\Lambda) \frac{\Delta t}{V} \sum F_{uncut}$$

Excess mass for redistribution:

$$V\Lambda(\hat{U} - U^*) = (1-\Lambda)\Lambda V(\tilde{U} - U^*)$$
$$= (1-\Lambda)\delta M$$

Cut Cell Redistribution



Excess mass is first distributed between the cutting grid and cut grid.

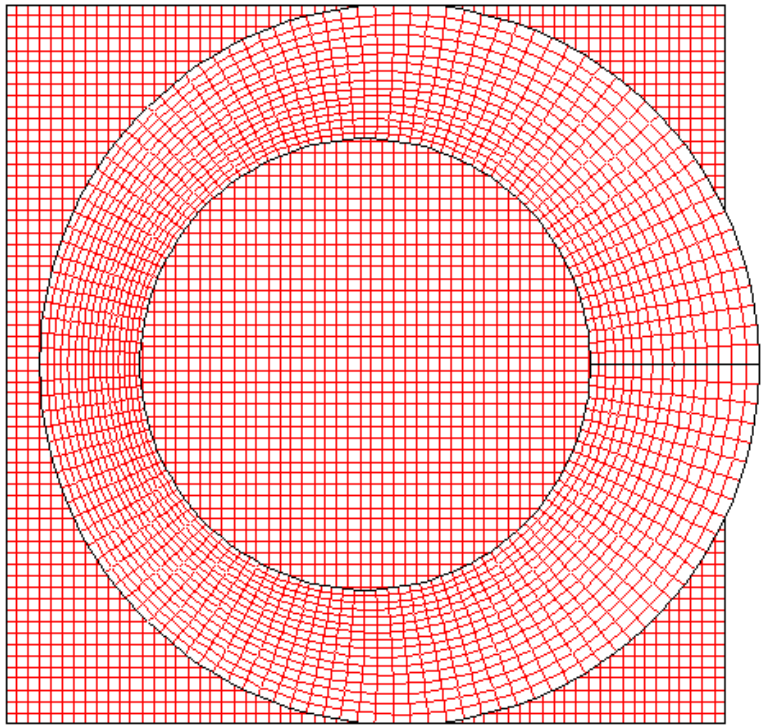
$$\theta = \left| \frac{\vec{u}}{|\vec{u}|} \bullet \vec{n} \right|$$

$$\delta M_{uncut} = \delta M * \theta$$

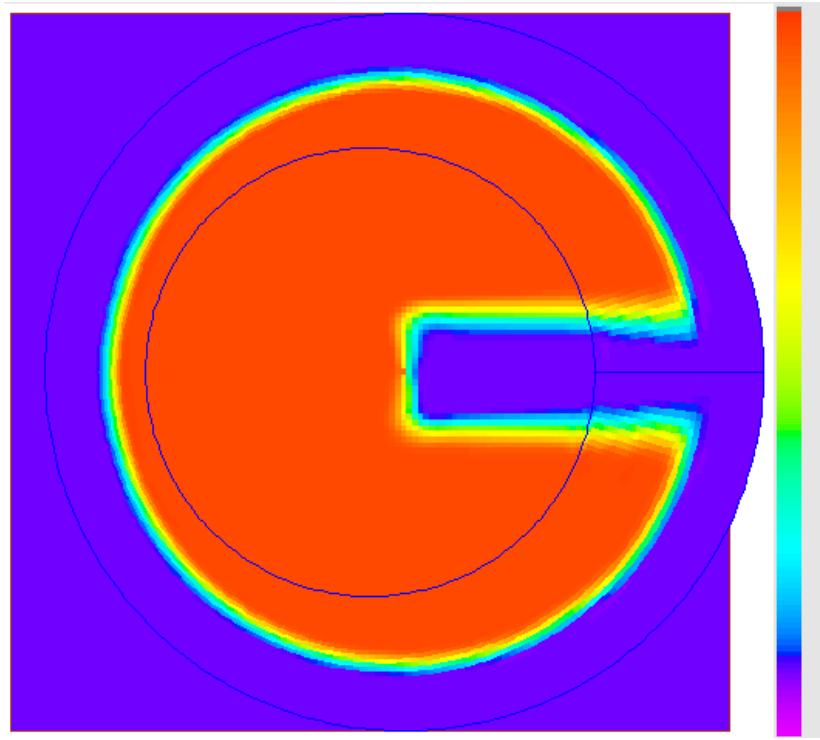
$$\delta M_{cut} = \delta M * (1 - \theta)$$

The distributed excess masses are next split between the cells using volume weighting.

Advection Test Problem



Computational Grid



Rotated Slot

Advection Convergence for Rotating Slot

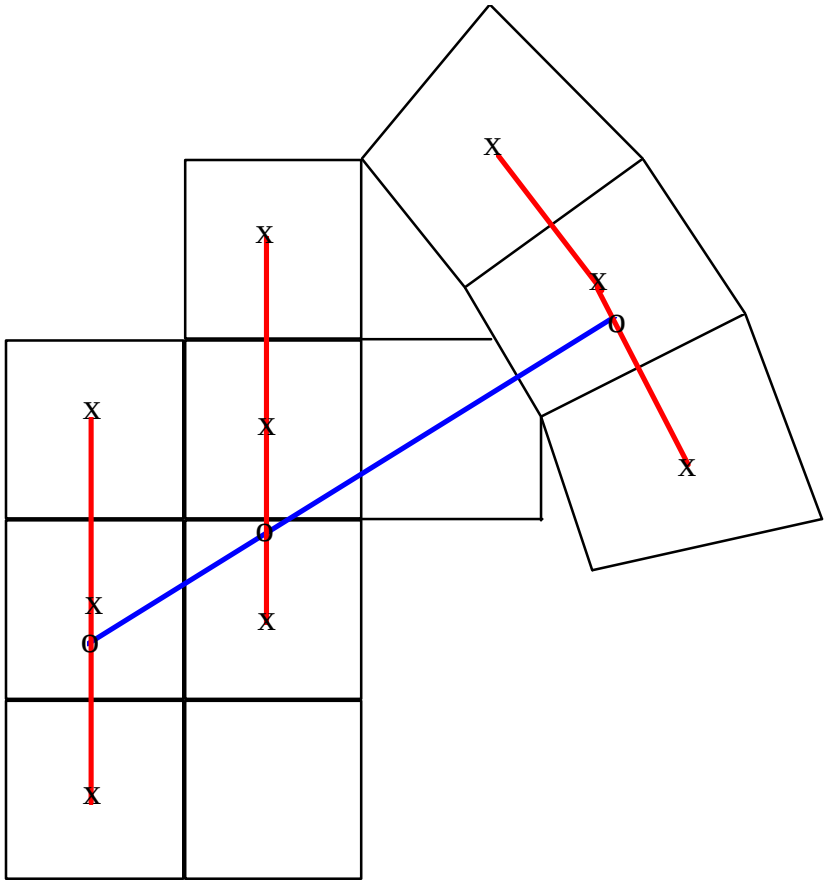
Linear Dim.	L_2 Error Sum	L_2 Error Norm	rate
33	1.404	.001034	
63	1.240	.000380	2.72
127	1.653	.000126	3.02
255	1.637	.000031	4.05

Elliptic Solver

- We have developed a discrete elliptic operator and multigrid hierarchy for solving the Laplace equation found in the projection algorithm.
- Properties:
  - 2nd order convergence in the max norm.
  - Uses a conservative formulation (div of divGrad operator is flux conservative).
  - Combines multigrid with a coarse level biconjugate gradient stabilized algorithm.

Discrete Gradient

- Gradients on regular portions of the grid return edge centered values using standard discretizations.
- Gradients on cut cells are found using 3rd order interpolation.



Discrete Divergence

- The divergence operator uses a finite volume representation for all cells.

$$\nabla \bullet u \approx \frac{1}{V}\int \nabla \bullet u \, dV = \frac{1}{V}\oint u \bullet d\vec{S}$$

- Using the trapezoid rule and second order fluxes, the operator is 1rst order at centroids of cut cells and 2nd order on regular cells.

Convergence Studies

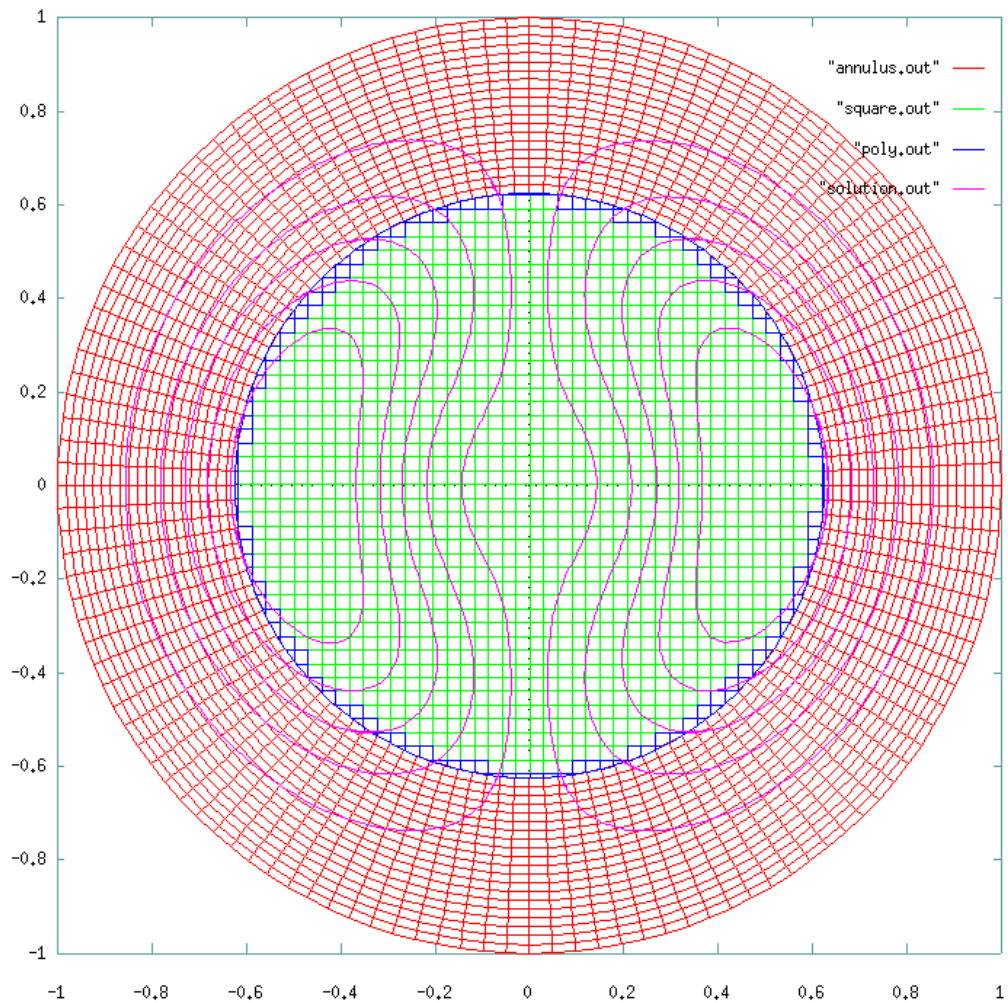
Maximum Error found for Analytic Solution

Ref. Factor	Max Norm	Rate
1	0.0238093	
2	0.0095272	2.499
4	0.0022597	4.216
8	0.0004874	4.636
16	0.0001268	3.844
32	0.0000336	3.774

Sample Solution

Converged Solution with RHS set so the solution has the form:

$$\varphi = \cos(\theta)\sin^3(\pi r)$$



Work in Progress

- Complete Projection Algorithm
- AMR Hybrid Mesh Elliptic Solver
- AMR Projection Algorithm
- 3-D AMR Algorithm
- General Grid Generation Code
  - Generate Cartesian cutout data structures for overset grids on a more general set of grids.
  - Build solver components for these data structures.
- General Grid AMR Incompressible Solver

Conclusion

- Proof of Principle shown for the components of an incompressible solver on a hybrid overset/Cartesian grid.
- Additional information available:
  - http://www.c3.lanl.gov/~jss
    - Presentations - this talk
    - Work in Progress