THE ELECTRICAL CONDUCTIVITY IN HIGH TEMPERATURE QED

EMIL MOTTOLA

Theoretical Division, T-8, MS B285, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
E-mail: emil@lanl.gov

LUIS M.A. BETTENCOURT

Theoretical Division, MS B288, Los Alamos National Laboratory, Los Alamos, NM 87545

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge MA 02139
E-mail: lmbett@lns.mit.edu

The calculation of the electrical conductivity of a high temperature QED plasma from first principles of QED is outlined. The principal feature is a non-trivial resummation of perturbation theory beyond hard thermal loops and truncation of the Schwinger-Dyson hierarchy including multiple scattering effects, consistent with gauge invariance.

Finite temperature quantum field theory has been well-studied in the past two decades in the imaginary time Matsubara formalism[1,2]. The imaginary time formalism is well suited for equilibrium thermodynamic quantities such as the free energy and pressure. More recently the real time-dependent non-equilibrium behavior of field theories has come to be investigated as well. These initial studies have been restricted to large N, or Hartree approximations for the most part. Such Gaussian approximations treat only the two-point function of the field theory or single particle distribution function in interaction with a time-dependent mean field, and ignore completely the direct scattering between the field quanta. The consistent inclusion of scattering processes in a practical real time formalism remains the principal challenge for future progress in non-equilibrium field theory.

The essential difficulty is that scattering requires higher connected Green's functions in the Schwinger-Dyson (SD) hierarchy of the field theory and it is not clear *a priori* how the infinite SD hierarchy should be truncated in a way that is both tractable and consistent with general principles of symmetry and renormalizability. Certainly simple perturbative expansion is not adequate, since the long time behavior of higher point Green's functions is generally non-perturbative. In fact, the most natural attempts to extend the large N

expansion beyond the leading order lead to secular instability in the Green's functions which is clearly unphysical. At the very least, scattering and self-energy effects must be resummed into the denominators of the two-point function to avoid such secular instabilities. This resummation problem problem becomes particularly acute in gauge theories where the Ward identities of the exact theory must be maintained in any resummation scheme.

The simplest and best understood gauge theory is electrodynamics, and the simplest non-equilibrium process is linear response, *i.e.* the small disturbance of the system and its relaxation back to equilibrium. If there is any non-equilibrium situation that should be under full theoretical control it is the linear response of a weakly coupled, infrared stable QED plasma. Conversely, if we cannot develop consistent theoretical methods which can handle this case, it is certain that non-Abelian plasmas far from equilibrium will remain completely beyond our abilities. It is remarkable that an apparently straightforward question like the calculation of the electrical conductivity of a relativistic QED e^+e^- plasma from first principles of the QED Lagrangian remains incompletely understood a half century after the consistent renormalizablity of QED was demonstrated. The reason of course is that one must confront the hierarchy and resummation problem in order to incorporate the scattering processes responsible for relaxation to thermal equilibrium and calculate the transport coefficients. In many-body theory it has long been recognized that extracting the hydrodynamic limit from microscopic degrees of freedom is highly non-trivial, even when those degrees of freedom are weakly coupled. The exact method by which irreversible behavior of collective modes emerges from fully reversible microscopic processes seems to depend on the details of the models considered and the approximations used. In addition, the technical issues of renormalizability and gauge invariance, typical of quantum field theory and not encountered in most many-body problems. Finally, the linear response theory of a QED plasma is already interesting in its own right, for comparison with known results for non-relativistic heterogeneous plasmas and for astrophysical applications. For example, the conductivity of the QED plasma is of vital importance to the understanding of the evolution of soft magnetic and electrical fields in the early universe.

With this set of motivations I will describe the background and recent progress on the first principles calculation of the electrical conductivity of a relativistic QED plasma. After outlining the general method of dealing with the SD hierarchy and gauge invariance, I will specialize to the ultra-relativistic case $T \gg m$ in order to extract analytic results which may be compared to other approaches. The work reported here is still in progress and a much more complete presentation of the results of our approach is now in preparation.

Consider first a very simple classical scattering model of electrical conductivity, due to Drude about a century ago. Let a medium consisting of essentially free charge carriers with mass m and charge e be subjected to an external electric field $\vec{E}$. Initially, the particles will be accelerated with

$$\vec{a} = \frac{e}{m}\vec{E}\,. \tag{1}$$

This acceleration is fully time reversible. Irreversibility enters by the explicit assumption that the particles scatter in a typical collisional time τ_c, after which they 'forget' their past acceleration. Then the average velocity of the charged particles in the medium is

$$\langle\vec{v}\rangle = \vec{a}\tau_c = \frac{e\tau_c}{m}\vec{E}\,. \tag{2}$$

If the number density of free charge carriers is n then the average electrical current in the medium is

$$\langle\vec{j}\rangle = en\langle\vec{v}\rangle = \frac{e^2 n\tau_c}{m}\vec{E}\,, \tag{3}$$

which is linear in $\vec{E}$. The coefficient of proportionality is the electrical conductivity,

$$\sigma_{DC} = \frac{e^2 n\tau_c}{m}\,, \tag{4}$$

where the subscript DC denotes the zero frequency, direct current limit.

This simple model already shows two essential features of a more complete approach. First, the transport coefficient is proportional to the collisional time scale τ_c. In the absence of collisions, τ_c and therefore σ_{DC} diverges, no matter how small the coupling e^2 is assumed. In field theory the effect of multiple collisions is contained in the imaginary parts of self-energies which must be computed accurately. Secondly, the conductivity is inversely proportional to the inertia of the charge carriers, m. This means that one must have a quasi-particle interpretation of the charge carriers with a well-defined real part of their self-energy. In other words, weak coupling implies that a narrow-width approximation to the spectral density of the charge carriers should be applicable. For an ultra-relativistic plasma the average inertia of these charge carriers is replaced by the temperature T, and the number density $n \propto T^3$. Hence the main issue is: what is τ_c and how is it to be computed in a self-consistent narrow-width approximation?

Naively one would expect this collisional time scale to be given by $(n\sigma_{sc})^{-1}$ where σ_{sc} is the two-particle scattering cross section obtained by the one-photon exchange diagram. Dimensionally the square of this diagram is proportional to α^2/T^2. Thus one would expect $\tau_c \propto (\alpha^2 T)^{-1}$ and $\sigma_{DC} \propto \alpha^{-1}T$. An analysis of the conductivity through the Boltzmann equation [4,6] suggests that $\sigma_{DC} \propto (\alpha \log \alpha)^{-1} T$, with the additional logarithm of the coupling coming about because of a logarithmic infrared divergence for small angle Coulomb scattering in the transport cross section, even when the finite range of the Coulomb interaction due to Debye screening effects in the plasma are taken into account. This residual sensitivity to small angle scattering and the appearance of logarithms of the coupling show the characteristic sensitivity of transport processes to soft physics and a ratio of hard to soft scales in the argument of the logarithm, in this case the Debye scale eT compared to the charge particle damping rate e^2T (up to additional logarithms which may be ignored to leading log order). Our principal interest will be in checking these estimates and the appearance of these scales in the microscopic equations of QED.

The starting point for the computation of the conductivity in QED is the linear response formula of Green-Kubo. The average current in an external potential can be expressed in the form,

$$\langle j_\mu(x) \rangle = -ie^2 \mathrm{Tr}\left[\gamma_\mu \mathcal{G}(x, x; A)\right] \tag{5}$$

where $\mathcal{G}(x, x'; A)$ is the full fermion propagator in the external A_μ. By varying this expression with respect to $A_\nu(x')$ and keeping only the term linear in the perturbing potential we obtain

$$\delta\langle j_\mu(x) \rangle = -\int dx' \Pi_{R\,\mu\nu}(x, x') \delta A^\nu(x') \,. \tag{6}$$

The real time polarization tensor $\Pi^{ab}_{\mu\nu}(x, x')$ is given by

$$\Pi^{ab}_{\mu\nu}(x, x') = ie^2 \int dz dz' \mathrm{Tr}\left\{(\gamma_\mu)_{cd;a} \mathcal{G}_{dd'}(x, z) (\Gamma_\nu)_{d'c';b}(z, z'; x') \mathcal{G}_{c'c}(z', x)\right\} \tag{7}$$

The (ab) indices are real time indices of the 2×2 CTP formalism. The retarded polarization tensor $\Pi_{R\,\mu\nu}$ is proportional to the sum of the (11) and (12) components of this 2×2 matrix. Expanding the real time sums over the repeated CTP indices on the right side gives three non-trivial terms involving the products $\mathcal{G}_R \mathcal{G}_R$, $\mathcal{G}_A \mathcal{G}_A$ and $\mathcal{G}_R \mathcal{G}_A$ respectively, where R and A denote retarded and advanced Greens's functions. The non-trivial vertex function,

$$(\Gamma_\nu)_{ab;c}(x, y; z) = \frac{\delta \mathcal{G}^{-1}_{ab}(x, y)}{\delta A^c_\mu(z)} \tag{8}$$

necessarily arises from the linear variation whenever $\mathcal{G}$ is non-trivial. Its apparently $2^3 = 8$ independent real time components can be shown to reduce to three independent (complex) components which appear in the three terms of Π_R. We remark also that the vertex function defined by this variation of the Green's function (8) is essential to demonstrating the conservation of the current and polarization operator.

Since the polarization operator is evaluated in zero external potential for linear response, it has all the symmetries of the unperturbed equilibrium state, which spacetime translationally invariant as well as rotationally invariant. Hence we can introduce the usual Fourier transform in space and (real) time. At finite temperature the invariances plus the conservation law obeyed by $\Pi_{\mu\nu}$ imply that it can be decomposed into a transverse and longitudinal part,

$$\Pi_{\mu\nu}(\omega;\vec{k}) = P^T_{\mu\nu}(\omega;\vec{k})\Pi_T(\omega,k) + P^L_{\mu\nu}(\omega;\vec{k})\Pi_L(\omega,k)\,. \tag{9}$$

The conductivity is defined as the time-irreversible (dissipative) response to a homogeneous and time-varying electric field, which is determined by the imaginary part of longitudinal polarization. The DC conductivity is then

$$\sigma_{DC} = \lim_{\omega\to 0} \frac{\mathrm{Im}\Pi_L(\omega,\vec{k}=0)}{\omega}\,. \tag{10}$$

Up until this point all formulas are completely general and exact.

It is this extreme infrared limit that makes the evaluation of the conductivity non-trivial in practice. For example if the bare vertex γ_μ is substituted for the exact vertex Γ_μ in (7), with either the bare or hard thermal loop Green's function used for $\mathcal{G}$, one quickly finds that the polarization tensor has a kinematic threshold, or cut starting at non-zero ω. Hence the value of the limit indicated in (10) is zero in this approximation. However, if one tries to add to this simple picture any finite number of self-energy insertions or photon line exchanges one encounters infrared divergences in the quantity σ_{DC} arising from the so-called 'pinching pole' singularities of the real time formalism. These arise from the mixed $\mathcal{G}_R\mathcal{G}_A$ terms in (7), since by causality their poles in the complex frequency plane are on opposite sides of the real axis. In the narrow-width approximation we are interested in because of weak coupling $\alpha \ll 1$, these pole singularities approach the real contour of integration from opposite sides and lead to a divergent result in the limit of zero fermion damping width.

Hence the imaginary part of the self energy of the e^+e^- fermions, Σ is essential to the problem and regulates the infrared divergences. This is consistent with our general classical considerations in the Drude model at the outset. This finite self-energy (for finite α) must appear in the propagator, which means that it adds to the inverse propagator,

$$\mathcal{G}^{-1}_{ab}(P) = G^{-1}_{ab}(P) + \Sigma_{ab}(P)\,, \tag{11}$$

where $G^{-1}(P) = i\gamma_\mu P^\mu$ is the bare inverse propagator. From (8) this immediately implies that the vertex must be non-trivial. In order not to bring in the infinite hierarchy of higher point Green's functions we must make some approximation to Σ that does not involve the vertex itself. The simplest possibility of using the bare Green's function G in Σ turns out not to work because there are still pinching poles in the final expression for σ_{DC}. The physical reason for this divergence is that $\mathcal{G}$, not G describes the physical quasi-particles, dressed by their multiple interactions with the plasma. Hence we must use the self-consistent quasi-particle approximation,

$$\Sigma_{ab}(P) = i \int \frac{d^4Q}{(2\pi)^4} (\gamma^\mu)_{aa';b} \mathcal{G}_{a'b'}(P+Q)(\gamma^\nu)_{b'b;c'}(D_{\nu\mu})_{c'c}(Q) \, , \tag{12}$$

where only the bare vertices having non-vanishing CTP components $(\gamma^\mu)_{11;1} = -(\gamma^\mu)_{22;2}$ appear.

For the photon Green's function $D_{\mu\nu}$ we must incorporate the effects of Debye screening or the long range Coulomb interaction will lead to infrared divergences in the scattering cross section and σ_{DC}. However, in the definition of this 'internal' polarization for defining $D_{\mu\nu}$ the question arises if one should use the dressed Green's function $\mathcal{G}$ or the bare one G. Although $\mathcal{G}$ would seem to be the safer 'more correct' choice, it is easy to see that the Ward identity,

$$Q^\mu D^{-1}_{\mu\nu}(Q) = 0 \tag{13}$$

is violated unless one also used a non-trivial vertex Γ_μ in the definition of $D^{-1}_{\mu\nu}$, which would then enter the self-energy and hence the variation of $\mathcal{G}^{-1}$ would involve the variation of this vertex and higher point functions of the Schwinger-Dyson hierarchy. The only way to avoid this infinite regression without violating Ward identities is to define

$$(D_{\mu\nu})^{-1}_{ab}(Q) = (d_{\mu\nu})^{-1}_{ab}(Q) - i \int \frac{d^4P}{(2\pi)^4} \mathrm{tr}\left\{(\gamma_\mu)_{cd;a} G_{dd'}(P+Q)(\gamma_\nu)_{d'c';b} G_{c'c}(P)\right\} \, , \tag{14}$$

with $(d_{\mu\nu})^{-1}_{11}(Q) = -(d_{\mu\nu})^{-1}_{22}(Q) = \delta_{\mu\nu}Q^2 - Q_\mu Q_\nu$ is the bare photon inverse propagator.

The last equation we need is the variation of $\mathcal{G}^{-1}$ with these approximations for the fermion and photon self-energies. Substituting (12) and (14) into (11) and (8) gives the non-trivial integral equation for the vertex,

$$\begin{aligned} \Gamma^\mu_{ab;c}(P, -P-K; K) = \gamma^\mu_{ab;c} - i \int \frac{d^4Q}{(2\pi)^4} \gamma^\alpha_{aa';d} \mathcal{G}_{a'a"}(P-Q) \times \\ \Gamma^\mu_{a"b";c}(P-Q, P+Q; K) \mathcal{G}_{b''b'}(-Q-P) \gamma^\beta_{b'b;d'}(D_{\alpha\beta})_{dd'}(Q) \, , \end{aligned} \tag{15}$$

corresponding to the resummation of the single dressed photon exchange, similar to that which appears in the Bethe-Salpeter equation.

These equations summarize the minimal approximation to the Schwinger-Dyson hierarchy that is necessary to control all the infrared divergences and extract a finite DC conductivity from the Green-Kubo formula. The three kinds of real time vertices which appear can be analyzed in the narrow-width approximation and a semi-analytic result obtained in terms of a series of finite temperature real-time integrations. A careful analysis of the self-energy shows that due to a cancellation of the lowest order effects one must work slightly harder to extract the leading non-zero order result for σ_{DC} and include also the 'crossed double rainbow' diagram in Σ with a corresponding additional crossed two-photon exchange diagram in Γ. A detailed analysis of this additional set of diagrams is now in progress. The comparison with the Boltzmann approach to σ_{DC} is also of some interest and will be presented in full elsewhere.

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